Cloud Attractors and Time-Inverted Julia Boundaries

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Cloud attractors are predicted to exist in simple invertible systems. The explicit noodle map (a three-dimensional diffeomorphism) and several 4-variable ODEs (describing chemical reactors) provide candidates. A "widely spaced" Lyapunov spectrum (+, $-\varepsilon$, –) may in general warrant a numerical search.

The first explicit example of a cloud attractor (in a 3-D *non*-invertible map) is due to Kaplan and Yorke [1]. The basic idea how to generate such attractors is due to Julia [2]. Julia realized that the locally self-similar basin boundaries he had found in 2-D noninvertible analytic maps [2] can be inverted by means of a trick, yielding attractors of the same shape. The trick is to switch randomly between the two nonunique inverses [2, 3]. Nonanalytic (real) 2-D maps with Julia-like boundaries – see [4] for a first proposed example – can obviously be inverted in the same way. Since a third (chaotic) variable is needed to generate the random switching, again a 3-D noninvertible map arises.

The very same principle has already been used to advantage by Barnsley and Harrington [5]. They inverted noninvertible real maps made up of several expanding pieces (piecewise-linear 2-D maps), obtaining maps consisting of several alternative contracting pieces to be subjected to the random switching. The resulting attractors can be tailor-made to match any preconceived self-similar fractal pattern ("collage theorem" [6]).

What is the explanation for the resulting shapes? Are there "strong" and "weak" versions to be distinguished? What is he connection to ordinary (invertible) dynamical systems?

Barnsley [6] already saw that the random switching can in principle be done by a third (chaos-generating) variable in such a way that the resulting overall 3-D map is invertible – a generalized Smale solenoid [7]. This presupposes that all radial cross-sections through the ring-shaped map are identified conceptually. The cloud thus only arises as a projection; no individual cross-section through the attractor displays a connected set. Apart from this "weak" possibility as it may be called, "strong" invertible cloud attractors must also exist – in 4-D invertible maps. Accordingly, weak cloud attractors can be expected to be found in 4-variable, and strong ones in 5-variable flows (like chemical ODEs). So far, not a single example of either kind has been seen, either in an explicit map or in a flow.

Very recently it was proposed that Julia boundaries arise whenever a hyperchaotic attractor is punctured by two or more adjacent basins [8]. According to this global-analytic interpretation, Julia boundaries are generalized "frontières floues" (fuzzy boundaries) in the sense of Mira [9]. (Mira had only considered the lower-dimensional, 2-D case; cf. also McDonald et al. [10] and Ling [17].) Julia boundaries therefore necessarily have a saddle structure in general (except in the noninvertible lower-dimensional limit considered by Julia). They therefore by definition should no longer yield attractors when time-inverted. This conclusion nevertheless is premature since chaotic attractors too involve unstable (saddle) structures without ceasing to be attractors.

What is the recipe to generate Julia attractors in invertible systems? Only the weak case (3-D maps/4-D flows) is to be considered in the following. The answer is: Prepare a noodle map.

There are two types of nontrivial invertible maps in 3 D, the folded-towel (or pancake) map, and the stretched stocking (or noodle) map [11]. The former has two positive Lyapunov characteristic exponents since there are two independent directions of repetitive stretching involved (hyperchaos [12]), the latter has only one and therefore at first sight appears to belong to the class of ordinary chaotic systems. This conclusion would be misleading, however, since the intrinsic complexity is the same: Every nontrivial noodle map becomes a folded-towel map under time inversion and vice versa [11]. The simplest example of a noodle map is the Smale solenoid [7], but with three rather than 2 windings of the first iterate inside. As it happens, Barnsley's solenoid [6] also belongs into this category – if more than two submaps take part in the switching.

If a doubly punctured folded-towel map generically yields a Julia boundary [8], this is tantamount to saying that a lengthwise contracting (time-inverted) noodle map, hidden in the expanding folded-towel map, generates a Julia boundary when made part of a basin boundary. Its temporal inverse, a lengthwise expanding (ordinary) noodle map, therefore generates a time-inverted Julia boundary. Can it be attracting?

It can only be attracting if the lengthwise expanding noodle map contains an attractor, a noodle-map attractor (see [13, 14] for examples). Unfortunately, the introduction of volume contraction renders the formation of sheet-like structures impossible in a crossection perpendicular to the expanding direction. On the one hand, there is only contraction left in two directions; on the other, the "skins" of 2 successive images (think of a covering initial box and its

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image) can never remain in touch in the interior of the preimage; this excludes even "perforated" tissue bridges there. Thus even with a knotted map (with infinitely many internal knots generated) everything will appear neatly separate locally in a cross section. Nevertheless there is a difference between "no entangling" and an entangling that "does its best" to cover the whole former separatrix even if not quite succeeding in doing so.

The condition is that the two negative exponents involved in generating the (in a cross-section) contracting noodle are not more or less equal (determining something like a spaghetti) but rather are markedly different ("flatness condition"). Specifically, cap-like coverings of one portion of the noodle by the other have to occur ("interlocking without touching"), in order then to be repeated at all resolutions within themselves again and again.

The present prediction will be easy to test numerically. A noodle diffeomorphism with constant Jacobian determinant is available [13] (also [14]) in which the two contracting components can be manipulated independently.

As to invertible flows an analogous prediction can be made. In such nonlinear systems (like 4-variable reaction-kinetic systems), frequently a spectrum of "widely spaced" Lyapunov characteristic exponents is found: one positive, one large negative, one zero, and one *small negative*; see [15, 16] for 2 examples. (Strangely, *two* positive Lyapunov exponents so far eluded detection in the same systems.) These systems apparently possess nontrivial noodle maps of the flat type for their cross sections. Again, detailed (if time-consuming) numerical studies of these attractors appear to be worth while.

To conclude, analogues to Julia boundaries occurring in invertible systems will form intriguing objects of study — and so will their attracting cousins.

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